International Journal of Applied Mathematics
& Statistical Sciences (IJAMSS)
ISSN(P): 2319-3972; ISSN(E): 2319-3980
Vol. 6, Issue 1, Dec – Jan 2017; 33-42

International Academy of Science, Engineering and Technology
Connecting Researchers; Nurturing Innovations

ESSENTIALLY DISTINGUISHED MODULES

LAYLA SALMAN MOHMOOD & SHAYMAA HABEEB HASAN

Department of Mathematics, College of Education IBN Al-Hai than for Pure Science, University of Baghdad, Iraq

ABSTRACT

Let R be a commutative ring with identity and M be a unitary (left) R-module. In this paper we present the concepts essentially distinguished module and essentially distinguished ring as generalizations of distinguished module and distinguished ring. M is called essentially distinguished R-module provided that $ann_M(I)$ is an essential sub module of M for each maximal ideal I of R. R is called R-distinguished ring if R as an R-module is distinguished. The basic properties of such modules (rings) are studied.

KEYWORDS: Distinguished Module, Distinguished Ring, Essentially Distinguished Module, Essentially Distinguished Ring, Co Generator Ring, Principally Quasi-Injective Module

1. INTRODUCTION

Throughout this paper all rings are commutative with identity. An R-module M is called distinguished if $ann_M(I) \neq 0$ for each maximal ideal I of R, [7]. The notion of distinguished module was introduced by G. Azumaya in [7] in establishing a theory of quasi-Frobenius module In 1996, L. S. Mahmood studied the concepts of distinguished module and distinguished ring, [9]. In this paper we introduce essentially distinguished module and essentially distinguished ring as a generalization of distinguished module and distinguished ring. It turns out that the class of essentially distinguished modules (rings) contains properly the class of distinguished modules (rings), and that the two classes are equivalent in certain classes of modules (rings). Many characterizations of such modules and rings are established in this work.

Notations: For an R-module M and an ideal I of R the set $ann_M(I) = \{ m \in M : am = 0 \; for \; all \; a \in I \}$ is the annihilator of I in M, $ann_R(I) = \{ r \in R : ra = 0 \; for \; all \; a \in I \}$ is the annihilator of I in R, and for $m \in M, ann_R(m) = \{ r \in R : rm = 0 \}$ is the annihilator of m in R.

2. ESSENTIALLY DISTINGUISHED MODULES

It is know that a sub module a non-zero sub module N of an R-module M is called essential if $N \cap K \neq 0$ for each non-zero sub module K of M, equivalently N is an essential sub module of M if for each $0 \neq m \in M$, there exists $r \in R$ such that $0 \neq rm \in N$, [5]. M is called a uniform module if every sub module of M is essential, [5]. We introduce the following concept:

Definition 2.1. An R-module M is called essentially distinguished if $ann_M(I)$ is an essential sub module of M (notation ally, $ann_M(I) \leq_e M$) for each maximal ideal I of R.

Remarks and Examples 2.2

1. An essentially distinguished module is distinguished but not conversely. For example Z_6 as a Z_6-module is

distinguished but not essentially distinguished since $ann_{Z_6}(\bar{2}) = (\bar{3}) \nleq_e Z_6$ and $ann_{Z_6}(\bar{3}) = (\bar{2}) \nleq_e Z_6$.

2. A uniform module is essentially distinguished if and only if it is distinguished.
3. Z_{12} as a Z_{12}-module is not essentially distinguished since $(\bar{2})$ is a maximal ideal in the ring Z_{12} but $ann_{Z_{12}}(\bar{2}) = (\bar{6}) \nleq_e Z_{12}$.
4. Z_{p^n} as a Z_{p^n}-module is essentially distinguished for all prime number *p* and positive integer *n*, Since the ideals of the ring Z_{p^n} are:

   $(\bar{p}) \supseteq (\overline{p^2}) \supseteq (\overline{p^3}) \supseteq \cdots \supseteq (\overline{p^{n-1}}) \supseteq (\overline{p^n}) = (\bar{0})$, but $(\bar{p})$ the only maximal ideal of Z_{p^n} and $ann_{Z_{p^n}}(\bar{p}) = (\overline{p^{n-1}}) \leq_e Z_{p^n}$.

5. A torsion-free module is not essentially distinguished; in fact it is not distinguished. For instance each of $Z, Q, Z\oplus Z, Z\oplus Q, \ldots$ as a *Z*-module is not essentially distinguished.
6. Z_{p^∞} as a *Z*-module is not distinguished and hence not essentially distinguished, for if $I = (q)$ be any maximal ideal of *Z* with *q* is a prime number, then $ann_{Z_{p^\infty}}(q) = Hom_Z(Z_q, Z_{P^\infty}) = \begin{cases} 0 \text{ if } p \neq q \\ Z_q \text{ if } p = q \end{cases}$.
7. An *R*-module *M* is essentially distinguished if and only if $ann_M(I) \leq_e M$ for each proper ideal *I* of *R*.
8. *M* is essentially distinguished *R*-module if and only if *M* is essentially distinguished $R/ann_R(M)$-module.

3. BASIC PROPERTIES OF ESSENTIALLY DISTINGUISHED MODULES

Proposition 3.1: Let $f: M \to \acute{M}$ be a monomorphism. If $\acute{M}$ is essentially distinguished then *M* is also essentially distinguished.

Proof: Let *I* be a maximal ideal of *R*. Then $ann_{\acute{M}}(I) \leq_e \acute{M}$. Let $K = ann_{\acute{M}}(I)$, so $K \leq_e \acute{M}$ and hence $f^{-1}(K) \leq_e M$ [6]. We claim that $ann_M(I) = f^{-1}(K)$. Let $x \in f^{-1}(K)$. Then $x \in M$ and $f(x) \in K$ and hence $f(x)I = 0 = f(xI)$, But f is a monomorphism implies that $xI = 0$, therefore $x \in ann_M(I)$. Hence $f^{-1}(K) \subseteq ann_M(I)$.

Now, If $x \in ann_M(I)$, then $xI = 0$ and $f(xI) = f(x)I = 0$ implies that $f(x) \in ann_{\acute{M}}(I) = K$, therefore $x \in f^{-1}(K)$. Thus $ann_M(I) \subseteq f^{-1}(K)$ So $ann_M(I) = f^{-1}(K) \leq_e M$ which completes the proof.

Corollary 3.2

1. Every non-zero sub module of an essentially distinguished module is also essentially distinguished.
2. Let $M_1 \simeq M_2$ be two *R*-modules. Then M_1 is essentially distinguished if and only if M_2 is so.

Lemma 3.3: Let M_1 and M_2 be two *R*-modules and *I* be an ideal of *R*. Then $ann_{M_1\oplus M_2}(I) = ann_{M_1}(I) \oplus ann_{M_2}(I)$.

Proof: Is straightforward and hence is omitted.

Proposition 3.4: Let M_1 and M_2 be two essentially distinguished *R*-modules. Then $M_1\oplus M_2$ is also essentially distinguished

Proof: Let I be a maximal ideal of R. Then $ann_{M_1}(I) \leq_e M_1$ and $ann_{M_2}(I) \leq_e M_2$. Therefore $ann_{M_1}(I) \oplus ann_{M_2}(I) \leq_e M_1 \oplus M_2$, [5]. According to lemma (3.3) we get $ann_{M_1 \oplus M_2}(I) \leq_e M_1 \oplus M_2$. Hence $M_1 \oplus M_2$ is essentially distinguished.

Corollary 3.5: The direct sum of a finite collection of essentially distinguished R-module is also essentially distinguished.

Proposition 3.6: Let M bea finitely generated R-module and S be a multiplicatively closed subset of R such that $I \cap S = \emptyset$ for each prime ideal I of R. Then M is essentially distinguished if and only if M_S is essentially distinguished R_S –module.

Proof: ($\Longrightarrow$) Let I_S be a prime ideal of R_S. Then I is a prime ideal of R[10]. Hence $ann_M(I) \neq 0$ (by hypothesis). But M isfinitely generatedimplies that $(ann_M(I))_S = ann_{M_S}(I_S)$[1]and hence $ann_{M_S}(I_S) \neq 0_S$[1]. On the other hand $ann_M(I) \leq_e M$ and according to [12],we get that $ann_{M_S}(I_S) \leq_e M_S$. Therefore M_S is essentially distinguished R_S-module.

($\Longleftarrow$) Let I be a prime ideal of R. Then I_S is a prime ideal of R_S[10], and by hypothesis we have$ann_{M_S}(I_S) \leq_e M_S$. Now, $ann_M(I))_S = ann_{M_S}(I_S)$ (since M is finitely generated), [1].Therefore $(ann_M(I))_S \leq_e R_S$andby [11],$ann_M(I) \leq_e R$, which completes the proof.

Corollary 3.7: Let M be a finitely generated R-module and p be a prime ideal of R. Then M is essentially distinguished R-module if and only if M_p is essentially distinguished R_p –module.

Now, we discuss essentially distinguishedness on a cogenerator ring (A ring R is called a cogenerator ring if the R-module R is a cogenerator for Mod-R, that is every R-module can be embedded in a direct product of copies of R).

Proposition 3.8: If R is a cogenertor ring, then every faithful uniform R-module is essentially distinguished.

Proof: Follow from the fact that every faithful module over a cogenerator ring is distinguished [9], and by (2.1,(2)) the result follows.

Corollary 3.9: If R is a quasi-Frobenuisring. Then every faithful uniform R-module is essentially distinguished.

Proof: R being quasi-Frobenius implies that R is a co generator ring [3]. Hence we get the result by proposition (3.8).

Corollary 3.10: If R is a co generator ring and M is a faithful R-module such that $E(M)$(the injective hull of M) is indecomposable then M is essentially distinguished.

Proof: $E(M)$ being indecomposable implies that M is uniform[8], and then by proposition (3.8),M is essentially distinguished.

Corollary 3.11: If R is a cogenerator ring and M is a faithful R-module which has exactly two closed submodules, then M is essentially distinguished.

Proof: As M has exactly two closed submodules implies that M is uniform [8], so M is essentially distinguished by proposition (3.8).

Corollary 3.12: If R is acogenertor ring and M is a faithful quasi-injective indecomposable R-module, then M is essentially distinguished.

Proof: Since M is quasi-injective and indecomposable implies that M is uniform [8]and hence essentially distinguished by proposition(3.8).

Corollary 3.13: If R is a cogenerator ring and M be a faithful indecomposable and extending R-module, then M is essentially distinguished.

Proof: M being extending and indecomposable gives M is uniform [15] and according to proposition (3.8), M is essentially distinguished.

Remark 3.14: The condition R is a cogenerator ring in proposition (3.8) and corollary (3.12) cannot be dropped. For instance Q as a Z –module is faithful and uniform, however it is not essentially distinguished in fact it is not distinguished, note that the ring Z is not a cogenerator ring. On the other hand$Q$ is an injective and hence quasi-injective Z-module, and that it is indecomposable.

4. SOME CHARACTERIZATIONS OF ESSENTIALLY DISTINGUISHED MODULE

Many interesting characterizations of essentially distinguished modules in certain classes of modules are given in this section.

Proposition 4.1: Let M be an R-module. If for each maximal ideal I of R, there exists $0 \neq m \in M$ such that$(m) \leq_e M$ and $I = ann_R(m)$, then$M$ is essentially distinguished.

Proof: Let I be a maximal ideal of R. By hypothesis $I = ann_R(\, m\,)$ for some $0 \neq m \in M$ and $(\, m\,) \leq_e M$. There for $Im = 0$ and hence $m \in ann_M(\, I\,)$ which implies that $(\, m\,) \leq ann_M(\, I\,)$. As $(\, m\,) \leq_e M$ gives$ann_M(\, I\,) \leq_e M$ [5]. Therefore M is essentially distinguished.

Remark 4.2 The condition $(\, m\,) \leq_e M$ in proposition (4.1) cannot be dropped. For example: Z_6as aZ_6-module is notan essentially distinguished. Note that$ann_{Z_6}(\bar{2}) = (\bar{3}) \nleq_e Z_6$ and $ann_{Z_6}(\bar{3}) = (\bar{2}) \nleq_e Z_6$.

Corollary 4.3: Let$M$be an R-module such that $(m) \leq_e M$ for each $0 \neq m \in M$. Then M is essentially distinguished if and only if M is distinguished.

In order to give a partial converse for proposition (4.1), the following areneeded

Lemma 4.4: Let M be an R-module such that $End_R(\, M\,) = S \simeq R$. If $Sm \subseteq Sn$ with $m, n \in M$ and $Sm = \{\, f(\, m\,) : f \in S\,\}$, then $Rm \subseteq Rn$.

Proof: Let$\varphi: R \to S$ be an isomorphism. For each$r \in R$, $\varphi(r) = \varphi_r: M \to M$ and $\varphi_r(m) = rm$ for all $m \in M$. Clearly $\varphi_r \in S$ and $\varphi_r(m) \in Sm$. Hence $\varphi_r(m) \in S_n$ (since $Sm \subseteq Sn$ by hypothesis) Therefore $\varphi_r(m) = f(n)$ for some $f \in S$, and hence $f = \varphi(t)$ for some $t \in R$ (since S ≃ R), on the other hand $\varphi(t) = \varphi_t$ gives $f(n) = \varphi_t(n)$ for all $n \in M$. So $rm = \varphi_r(m) = \varphi_t(n) = tn$ for all $r \in R$.**i**mplies thatRm ⊆ Rn

Recall that an R-module M is called principally injective if each R-homomorphism $\alpha : Ra \longrightarrow M$ such that $a \in R$, extends to R , [14].And M is called principally quasi- injective if for each $m \in M$ each homomorphism $f: Rm \to M$ can be extended to an endomorphism of M, [13].

Proposition 4.5[13]: Let M be an R-module and $S = End_R(M)$. Then the following statements are equivalent:

1. M is principally quasi- injective.
2. $ann_M(\, ann_R(\, m\,)) = S_m$ For all $m \in M$.
3. If $ann_R(m) \subseteq ann_R(n)$ where $m, n \in M$, then $S_n \subseteq S_m$.
4. For each $m \in M$, if $\alpha, \beta: Rm \to M$ with β is a monomorphism, then there exists $\delta \in S$ such that $\delta \circ \beta = \alpha$.

Now, we present a partial converse for proposition (4.1).

Proposition 4.6: Let M be a principally quasi-injective R-module and $End_R(M) = S \simeq R$. If M is essentially distinguished then for each maximal ideal I of R there exists $m \in M$ such that $(m) \leq_e M$ and $I = ann_R(m)$.

Proof: Let I be a maximal ideal of R. Then by [11], there exists $0 \neq m \in M$ such that $I = ann_R(m)$. It is left to show that $(m) \leq_e M$. Let $0 \neq x \in M$. As M is essentially distinguished, then $ann_M(\, I\,) \leq_e M$, therefore there exists $r \in R$ such that $0 \neq rx \in ann_M(\, I\,)$. Hence $Irx = 0$ implies that $I \subseteq ann_R(\, rx\,)$, therefore $ann_R(m) \subseteq ann_R(rx)$ and according to proposition (4.5,(3)) we get $Srx \subseteq Sm$ and by lemma (4.4) implies that $Rrx \subseteq Rm$, therefore $rx \in (m)$ Whichis what we wanted.

Theorem 4.7: Let M be a principally quasi-injective R-module such that $End_R(\, M\,) \simeq R$.Then M is essentially distinguished if and only if for each maximal ideal I of R, there exists $0 \neq m \in M$ such that $(m) \leq_e M$ and $I = ann_R(m)$.

Proof: Follows from propositions (4.1) and (4.6).

Corollary 4.8: Let M be a faithful scalar and principally quasi-injective R-module. Then M is essentially distinguished if and only if for each maximal ideal I of R there exists $m \in M$ such that $(m) \leq_e M$ and $I = ann_R(m)$.

Proof: M being faithful R-module gives $End_R(M) \simeq R$[4]. Hence the result follows from theorem (4.7).

Proposition 4.9: If M is an essentially distinguished R-module, then for each maximal ideal I of R, $ann_M(\, I\,)$ contains a copy of every simple R-module.

Proof: Let P be a simple R-module. Then $P \simeq R/I$ for some maximal ideal I of R. M is distinguished gives that M contains a copy of P [9]. So there exists a monomorphism say $f: P \to M$. Let $0 \neq x \in P$ and put $f(x) = m$, with $0 \neq m \in M$. But $ann_M(\, I\,) \leq_e M$ implies there exists $r \in R$ such that $0 \neq rm \in ann_M(\, I\,)$, $r\, m = rf(x) = f(\, rx\,) \in ann_M(\, I\,)$. hence $0 = If(\, rx\,) = f(\, Irx\,)$, therefore $Irx = 0$, gives $rx \in ann_M(\, I\,)$.Hence $(0) \neq (\, rx\,) \subseteq ann_M(\, I\,)$. On the other hand $(\, rx\,) \leq P$ and P is simple implies that $P = (\, rx\,) \subseteq ann_M(\, I\,)$ which completes the proof.

A partial converse of proposition (4.9) is establish in the following proposition.

Proposition 4.10: Let M be a scalar and principally quasi-injective R-module. If for each maximal ideal I of R, $ann_M(\, I\,)$ contains a copy of every simple R-module, then M is essentially distinguished.

Proof: Let I be a maximal ideal of R. By hypothesis, $ann_M(\, I\,)$ contains a copy of R/I, so $ann_M(\, I\,) \neq 0$ and hence M is distinguished. We have to show that $ann_M(\, I\,) \leq_e M$. Let $0 \neq x \in ann_M(\, I\,)$ and let A be a maximal submodule of Rx. Then Rx/A is a simple R-module Let $f: Rx/A \to ann_M\, (\, I\,)$ be a monomorphism. Define $g: Rx \to M$ by $g(m) = f(m + A)$ for all $m \in Rx$. As M is principally quasi-injective implies that there exists a homomorphism $h: M \to$

M such that $h \circ i = g$. Where $i: Rx \to M$ is the inclusion homomorphism. On the other hand M is a scalar R-module implies that there exists $t \in R$ such that $h(m) = tm$ for all $m \in M$[2]. Therefor for all $m \in Rx$, $g(m) = f(m + A) = h(m) = tm$. Thus $tx = f(x + A) \neq 0$. For if $f(x + A) = 0$ gives $x + A = A$ and hence$Rx = A$ which is a contradiction.

Now, $tIx = g(Ix) = f(Ix + A) = If(x + A) = 0$. So, $tIx = 0 = Itx$ implies that $tx \in ann_M(\ I\ )$ and hence $ann_M(\ I\ ) \leq_e M$.

Corollary 4.11: Let *M* be a cyclic and principally quasi-injective *R*-module. Then *M* is essentially distinguished if and only if for each maximal ideal *I* of *R*, $ann_M(\ I\ )$contains a copy of every simple *R*-module.

Proof: Follows from the two proposition 4.9, 4.10 and from the fact that every cyclic module is a scalar module [2].

In the following proposition we introduce a necessary condition for essentially distinguishedness by using the concept of injective hull of a module.

Proposition 4.12: Let *M* be an essentially distinguished *R*-module. Then$E(ann_M(I))$ (the injective hull of $ann_M(I)$)is a cogenerator for Mod-*R* for each maximal ideal *I* of *R*.

Proof: Let I be a maximal ideal of *R*. Then $ann_M(\ I\ )$ contains a copy of every simple *R*-module(by proposition4.10). But$ann_M(\ I\ ) \subseteq E(ann_M(\ I))$, therefore$E\big(ann_M(\ I\ )\big)$ contains a copy of every simple *R*-module, and since $E(ann_M(\ I\ ))$ is an injective *R*-module, therefore $E(ann_M(\ I\ ))$ is a cogenerator for Mod-*R*.[6]

Corollary 4.13: If *M* is an essentially distinguished *R*-module, then *E(M)* is a cogenerator for Mod-*R*.

Proof: Let *I* be a maximal ideal of *R*. Then$ann_M(\ I\ ) \leq_e M$and hence$E\big(ann_M(\ I\ )\big) = E(M)$[12], therefore *E(M)* is a cogenerator for Mod-*R* proposition 4.12.

Corollary 4.14: Let *M* be an essentially distinguished *R*-module, and*I* be a maximal ideal of *R*. If$ann_M(I)$ is an injective*R*-module, then$ann_M(I)$is acogenerator for Mod-*R*.

Proof: Follows from proposition (4.12) and from the fact that $E(M) = M$ if and only if *M* is injective [6].

A partial converse of proposition (4.12) is provided proposition by the following proposition:

Proposition 4.15: Let *M* be a scalar (or a cyclic) principally quasi-injective *R*-module. If for each maximal ideal *I* of *R*, $E(ann_M(\ I\ ))$ is compressible and a cogenerator for Mod-*R*, then *M* is essentially distinguished

Proof: Let*I* be a maximal ideal of *R*. By hypothesis $E\big(ann_M(\ I\ )\big)$ is a cogenerator for Mod-*R*, on the other hand it is an injective R-module implies that$E(ann_M(\ I\ ))$ contains a copy of every simple R-module [12]. But $E\big(ann_M(\ I\ )\big)$ is compressible and $ann_M(\ I\ ) \leq E(ann_M(\ I\ )$ implies that there exists a monomorphism say $f: E(ann_M(\ I\ )) \to ann_M(\ I\ )$. Therefore$ann_M(\ I\ )$ contains a copy of every simple *R*-module and according to proposition (4.10)we get *M* is essentially distinguished.

Corollary 4.16: Let *M* be a scalar (or acyclic) principally quasi-injective *R*-module and I be a maximal ideal of *R*. If$ann_M(\ I\ )$ is a cogenerator for Mod-*R* and $E(ann_M(\ I\ ))$ is compressible, then *M* is essentially distinguished.

Proof: Let I be a maximal ideal of R. By hypothesis, $ann_M(\ I\ )$ is a cogenerator for Mod-*R*, therefore

$E(ann_M(\ I\ ))$ is also a cogenerator for Mod-R, [12], and hence the result follows by proposition (4.15).

Proposition 4.17: If M is an essentially distinguished R-module and I is a maximal ideal of R, then every finitely generated(projective or multiplication) R-module is dualizable with respect to $ann_M(\ I\ )$.

Proof: Let N be a finitely generated R-module and Let K be a maximal submodule of N. Then N/K is a simple R-module. But M is essentially distinguished implies that there exists a monomorphism say $f: N/K \to ann_M(\ I\ )$ (by proposition 4.10), and hence $f\pi \in Hom(N, ann_M(\ I\ ))$where $\pi: N \to N/K$ is the natural homomorphism. If $f\pi = 0$, then $0 = f\pi(N) = f(N/K\ )$ implies that $N/K = \bar{0}$ and hence $K = N$ which is a contradiction.

Proposition 4.18: Let M be a scalar (a cyclic) principally quasi-injective R-module and I be a maximal ideal of R. If every finitely generated (projective or multiplication) R-moduleisdualizable with respect to$ann_M(\ I\ )$, then M is essentially distinguished.

Proof: Let p be a simple R-module. Then $P \simeq R/I$ for some maximal ideal I of R. By hypothesis $Hom_R(P, ann_M(\ I\ )) \neq 0$. Let $f: P \to ann_M(\ I\ )$ bea non-trivial homomorphism. Then f is a monomorphism. Therefore $ann_M(\ I\ )$ contains a copy of P which implies that$ann_M(\ I\ ) \neq 0$. Next we proceed as in the proof of proposition (4.10) to prove that$ann_M(\ I\ ) \leq_e M$.

We can summarize the characterizations of essentially distinguishedness as in the following theorem.

Theorem 4.19: Let M be a scalar (cyclic) principally quasi-injective R-module and let I be a maximal ideal of R. Then the following statements are equivalent:

- M is essentially distinguished.
- $ann_M(\ I\ )$ Contains a copy of every simple R-module.
- $E(ann_M(\ I\ ))$Is a cogenerator for Mod-R Provided that $E(ann_M(\ I\ ))$is compressible.
- Every finitely generated (or projective or multiplication) R-module is dualizable with respect to $ann_M(\ I\ )$.

5. ESSENTIALLY DISTINGUISHED RINGS

In this section we consider the rings R for which the R-module R is essentially distinguished.

Definition 5.1: A ring R is called essentially distinguished if $ann_R(\ I\ ) \leq_e R$ for each maximal ideal I of R.

Remarks and Examples 5.2

1. Every essentially distinguished ring is distinguished but the converse is not true in general. The following example shows: A ring Z_6 is distinguished but not essentially distinguished.
2. If p is a prime number and n is a positive integer then the ring Z_{p^n} is an essentially distinguished ring.

   Proof: As in the proof of (2.2,4).
3. Every field is an essentially distinguished ring but the converse is not true in general, for instance, Z_4 is an essentially distinguished ring which is not a field.
4. An integral domain which is not a field is not essentially distinguished (in fact not distinguished).

5. A ring R is essentially distinguished if and only if for each proper ideal I of R, $ann_R(I) \leq_e R$.

   Proof: As in the proof (2.2, 7).

6. Let $f: R_1 \to R_2$ be a ring monomorphism. If R_2 is essentially distinguished, then R_1 is also essentially distinguished.

   Proof: Is similar to the proof of proposition (3.1).

7. Every subring of essentially distinguished ring is also essentially distinguished.
8. The direct sum of two (a finite family) essentially distinguished rings is essentially distinguished.

   Proof: Is similar to the proof of proposition (3.4).

Proposition 5.3: Let R be a ring. If for each maximal ideal I of R there exists $0 \neq a \in R$ such that $(a) \leq_e R$ and $I = ann_R(a)$, then R is essentially distinguished.

Proof: Is similar to the proof of proposition (4.1).

Corollary 5.4: Let R be a ring such that $(a) \leq_e R$ for each $a \in R$. Then R is essentially distinguished if and only if R is distinguished.

The following result is a partial converse for proposition (5.3).

Proposition 5.5: Let R be a cogenerator ring. If R is essentially distinguished, then for each maximal ideal I of R, there exists $0 \neq a \in R$ such that $(a) \leq_e R$ and $I = ann_R(a)$.

Proof: Let I be a maximal ideal of R. By [11],$I = ann_R(a)$ for some $0 \neq a \in R$ We have to show that $(a) \leq_e R$. Let$0 \neq t \in R$, then there exists $0 \neq r \in R$ such that$0 \neq rt \in ann_R(I)$. Hence$rt \in ann_R(ann_R(a))$, but R is a co generator ring implies that $ann_R(ann_R(a)) = (a)$[6], so $rt \in (a)$ and hence$(a) \leq_e R$.

For the next result the following lemma is needed.

Lemma 5.6: Let M be a finitely generated faithful multiplication R-module and I be a non-zero ideal of R. Then $I \leq_e R$ if and only if $IM \leq_e M$.

Proof: Suppose that$I \leq_e R$. Let $N \leq M$ and $N \cap IM = (0)$. As M is a multiplication R-module, then $N = JM$ for some ideal J of R So, $(0) = JM \cap IM = (J \cap I)\, M$ (Since is M faithful multiplication), implies that $J \cap I = 0$ which is a contradiction. There for $IM \leq_e M$.

Conversely**,** suppose that $IM \leq_e M$. Let L be an ideal of R and $L \cap I = (0)$. Then $(0) = (L \cap I)M = LM \cap IM$ (Since M is faithful multiplication). But $IM \leq_e M$, therefore a contradiction. Hence $I \leq_e R$.

The relationship between essentially distinguished ring and essentially distinguished module in the class of multiplication modules is established in the following theorem:

Theorem 5.7: Let M be a finitely generated faithful multiplication R-module. Then M is essentially distinguished if and only if R is essentially distinguished ring.

Proof: ($\Longrightarrow$)If M is essentially distinguished, let I be a maximal ideal of R. Then $ann_M(I) \leq_e M$. Let $N = ann_M(I)$. Then $N = JM$ for some non-zero ideal J of R, and by lemma (5.6), $J \leq_e R$. On the other hend it can be easily checked that $ann_M(I) = ann_R(I)M$ which gives that$ann_R(I) \leq_e R$by lemma (5.6), and hence R is essentially distinguished.

(⟸) Suppose that R is essentially distinguished ring. Let I be a maximal ideal of R. Then $ann_R(I) \leq_e R$ and hence $ann_R(I)M \leq_e M$ (by lemma (5.6)). But $ann_M(I) = ann_R(I)M$, therefore $ann_M(I) \leq_e M$ and hence M is essentially distinguished.

Corollary 5.8: Let M be a cyclic faithful R-module. Then M is essentially distinguished if and only if R is essentially distinguished.

REFERENCES

1. Atiyah M. F. and Macdonald I. C. Introduction to Commutative Algebra. Addison-Wesley publishing Company, 1969.
2. B. N. Shihab, Scalar Reflexive Modules, ph. D. Thesis, College of Education Ibn Al-Haithem, University of Bagdad, 2004.
3. C. Faith and E. A. Walker, "Direct Sum Representation of Injective modules", G. of Algebra 5,(1979), 203-221.
4. E. A. Al-Mohammed Ali, On Ikeda-Nakayoma Modules, ph. D. Thesis, College of Education Ibn Al-Haitham, University of Baghdad, 2006.
5. F. Kasch, Modules and Rings, Academic Inc. Press, London (1982).
6. F. W. Anderson, and K. R. Fuller, Rings and Categories of Modules, Springer-Verilog, New York, (1992).
7. G. Azumaya, a Duality Theory for Injective Modules, American. G. Math. (81)1959, 249-278.
8. K. R. Goodearl, Ring Theory, Nonsingular Rings and Modules**,** Marcel Dekker, Inc. New York and Basel. (1976).
9. L. S. Mahmood, Quasi-Frobenius Modules and Distinguished Modules, ph. D. Thesis, University of Baghdad, 1996.
10. M. D. Larsen, P. J. McCarthy, Multiplicative Theory of Ideals**.** Academic press, New York and London, 1971.
11. S. H. Hasan, Some Generalizations of Distinguished Modules, ph. D. Thesis, University of Baghdad, 2016.
12. T. Y. Lam, Lectures on Modules and Rings, Berkeley, California springer, (1998).
13. W. K. Nicholson, J. K. Park and M. F. Youssef, Principally quasi-injectivemodules, comm. Algebra, 27:4(1999), 1683-1693.
14. W. K. Nicholson and M. F. Yousif, Principally injective rings, J. Algebra, 174(1995),77-93.
15. Z. T. S. Al-Zubaidey. On purely Extending Modules, M. Sc. Thesis University of Baghdad, 2005.